Analytic Formulae for Gravity Turns

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A "gravity turn", more accurately a *zero angle-of-attack trajectory*, is a rocket trajectory where the axis of the rocket remains aligned with its velocity vector through the air. As gravity bends the trajectory, the vehicle turns to face the new vector. This ensures that thrust and aerodynamic forces are always along the axis of the vehicle, which is good for stability and structural integrity.

Gravity turns may be used for both ascent and descent trajectories. They may even be employed around airless bodies: in that case the "air" velocity is simply the velocity with respect to the surface. A gravity turn is not strictly the most efficient ascent/descent profile for an airless body, but is a handy compromise between gaining/shedding both horizontal speed and height above ground, as might be needed for clearing terrain.

Basic Equations

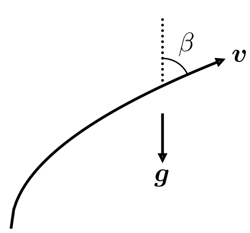
The basic law of motion is of course Newton's law:

$$m\dot{\boldsymbol{v}} = \boldsymbol{F}_{\text{thrust}} + \boldsymbol{F}_{\text{drag}} + \boldsymbol{F}_{\text{gravity}} \tag{1}$$

The defining feature of a gravity turn is $\boldsymbol{F}_{\text{thrust}} \parallel \boldsymbol{F}_{\text{drag}} \parallel \boldsymbol{v}$: only $\boldsymbol{F}_{\text{gravity}} = m\boldsymbol{g}$ provides any turning force. We define the *effective thrust-to-weight ratio* n as:

$$n \equiv \frac{F_{\text{thrust}} \pm F_{\text{drag}}}{mg} \tag{2}$$

choosing the $-$ sign when drag opposes thrust (ascent) and $+$ when they act in concert (descent). Without loss of generality we will assume ascent trajectories; descent trajectories are their mirror-image (only changing the value of n). Letting β be the angle between $\boldsymbol{v}$ and vertical (i.e. between $\boldsymbol{v}$ and $\boldsymbol{g}$), the components of Equation (1) along and transverse to $\boldsymbol{v}$ give us;



$$\dot{v} = gn - g\cos\beta \tag{3}$$
$$v\dot{\beta} = g\sin\beta \tag{4}$$

Realisticly these must be computed numerically, since n changes in a complicated manner depending on engine throttling, fuel consumption, drag forces, etc. We also ignore the potentially significant curvature of the planetary surface, changing $\boldsymbol{g}$, etc. The following section simplifies to the case of *constant* n, which can be treated as an *approximation* if n changes only slightly or a suitable average $\bar{n}$ is used, and explicitly assumes flat surfaces and uniform gravity.

Constant Thrust-to-Weight Approximation

For constant n, we could divide Equations (3) and (4) to get

$$\frac{1}{v}\frac{dv}{d\beta} = \frac{n - \cos\beta}{\sin\beta}$$

which we could integrate directly to give $v(\beta)$. However, we follow the derivation found in "Analytic Derivation of Ascent Trajectories and Performance of Launch Vehicles" by Paolo Teofilatto, Stefano Carletta, and Mauro Pontani (*Appl. Sci.* **2022**, 12, 5685), and transform to a new angular coordinate:

$$z = \tan\tfrac{\beta}{2}\,, \qquad \cos\beta = \frac{1-z^2}{1+z^2}\,, \qquad \sin\beta = \frac{2z}{1+z^2} \tag{5}$$

Equations (3) and (4) then become:

$$\dot{v} = gn - g\frac{1-z^2}{1+z^2} \tag{6}$$

$$v\dot{z} = gz \tag{7}$$

which we divide and rearrange to get:

$$\frac{dv}{v} = \left(n - \frac{1-z^2}{1+z^2}\right)\frac{dz}{z}$$

and then integrate to obtain:

$$v(z) = Az^{n-1}(1+z^2) \tag{8}$$

where A is a constant of integration.

The nature of a gravity turn is that it always asymptotes to $z \to 0$ (vertical) as $v \to 0$. Thus for launches, A is not set by initial condition *per se*, but selects among trajectories with different tip rates *after* launch. It can be chosen based on some final target tip angle and speed, then used to make small corrections to the trajectory early on to give the required tip rate. For descents, A can simply be determined from the initial speed and direction at the start of the turn: see *Descent*, below.

The actual trajectory, i.e. the height h and downrange distance s, can be found through a similar procedure:

$$\dot{h} = v\cos\beta = v\frac{1-z^2}{1+z^2}\,, \qquad \dot{s} = v\sin\beta = v\frac{2z}{1+z^2}$$

$$\frac{dh}{dz} = \frac{\dot{h}}{\dot{z}} = \frac{v^2}{gz}\frac{1-z^2}{1+z^2} \qquad \frac{ds}{dz} = \frac{\dot{s}}{\dot{z}} = \frac{v^2}{gz}\frac{2z}{1+z^2}$$

Substituting Equation (8) then gives:

$$\frac{dh}{dz} = \frac{A^2}{g}(z^{2n-3} - z^{2n+1}) \qquad\qquad \frac{ds}{dz} = \frac{2A^2}{g}(z^{2n-1} + z^{2n+1})$$

which integrate to:

$$h(z) = h_0 + \frac{A^2}{g}\left(\frac{z^{2n-2}}{2n-2} - \frac{z^{2n+2}}{2n+2}\right) \tag{9}$$

$$s(z) = s_0 + \frac{2A^2}{g}\left(\frac{z^{2n-1}}{2n-1} - \frac{z^{2n+1}}{2n+1}\right) \tag{10}$$

Equation (9) above is exactly equation (14) from Teofilatto *et al.* Equation (10) above corrects an error in equation (15) of Teofilatto *et al.* Without loss of generality we will treat h and s as vertical and horizontal *displacements* by setting $h_0 = s_0 = 0$.

Descent

For a gravity-turn *descent*, we start at some height h with some speed v and vertical angle parameter z, and would like to find thrust-to-weight ratio n required to land safely. In particular, we would like to begin our burn when n is near our engine's (presumed constant) maximum thrust-to-weight ratio, and maintain that thrust to landing. It may seem odd that n appears as an exponent in the expression for h, but it simplifies once we eliminate the constant A by substituting Equation (8):

$$\begin{aligned} h &= \frac{A^2}{g}\left(\frac{z^{2n-2}}{2n-2} - \frac{z^{2n+2}}{2n+2}\right) \\ &= \frac{v^2}{2g}\frac{1}{(1+z^2)^2}\left(\frac{1}{n-1} - \frac{z^4}{n+1}\right) \\ \Rightarrow \qquad 0 &= n^2 - \frac{v^2}{2gh}\frac{1-z^4}{(1+z^2)^2}n - \left(\frac{v^2}{2gh}\frac{1+z^4}{(1+z^2)^2} + 1\right) \end{aligned}$$

For descent, we have to either reverse the meaning of β or invert z. We do the latter, setting $z = \cot\frac{\beta}{2}$ and applying trig identities to obtain:

$$0 = n^2 + \left(\frac{v^2}{2gh}\cos\beta\right)n - \left(\frac{v^2}{4gh}(1+\cos^2\beta) + 1\right)$$

whence our solution:

$$n = -\frac{v^2}{4gh}\cos\beta + \sqrt{\left(\frac{v^2}{4gh}\cos\beta\right)^2 + \frac{v^2}{4gh}(1+\cos^2\beta) + 1}$$

which we write more simply as:

$$n = -\alpha\cos\beta + \sqrt{\alpha(\alpha+1)\cos^2\beta + \alpha + 1}\,, \qquad \alpha \equiv \frac{v^2}{4gh} \tag{11}$$

Thus by keeping track of v, h, and β, one waits until the moment when n equals the engine's maximum thrust-to-weight ratio (less some safety margin), then throttles the engine to maintain that n for the rest of the descent. This should end with the rocket having $v \to 0$ and $\beta \to 0$ at $h = 0$: an ideal touchdown.

Downrange Distance

Once thrust begins, the remaining downrange distance can be computed given z and n:

$$\begin{aligned}
s &= \frac{2A^2}{g}\left(\frac{z^{2n-1}}{2n-1} + \frac{z^{2n+1}}{2n+1}\right) \\
&= \frac{2v^2}{g}\frac{1}{(1+z^2)^2}\left(\frac{z}{2n-1} + \frac{z^3}{2n+1}\right) \\
&= \frac{v^2}{g}\frac{2z}{1+z^2}\frac{1}{(2n-1)(2n+1)}\left(2n + \frac{1-z^2}{1+z^2}\right) \\
&= \frac{v^2}{4g}\frac{\sin\beta\,(2n-\cos\beta)}{n^2-1/4}
\end{aligned}$$

where we have again used the inverted $z = \cot\frac{\beta}{2}$. Thence:

$$s = \alpha h \frac{\sin\beta\,(2n-\cos\beta)}{n^2-1/4} \qquad (12)$$

If there is non-flat topography, the elevation of the ground at a distance s can be tracked (with elevation maps or forward-looking radar) and fed in to the determination of h, and hence n, while waiting for the gravity turn burn to start. Once the burn starts the predicted landing spot should not change so long as one maintains constant n.

Adjusting trajectories to pick and choose a particular landing spot is beyond the scope of this document.

Time to Landing

Substituting Equation (8) into (7), we obtain:

$$\frac{dz}{dt} = \frac{g}{Az^{n-2}(1+z^2)}$$

which we can integrate to obtain:

$$\Delta t = \frac{A}{g}\left(\frac{z^{n-1}}{n-1} + \frac{z^{n+1}}{n+1}\right) \qquad (13)$$

Back-substituting Equation (8) and taking $z = \cot\frac{\beta}{2}$ for descending trajectories, we find an expression for time to landing:

$$\Delta t = \frac{v}{g}\frac{n-\cos\beta}{n^2-1} \qquad (14)$$

Gravity Drag

Now let's return to an ascent trajectory. Let us consider a gravity turn that ends with a horizontal trajectory with speed v_f. Since the final trajectory is horizontal, $z_f = \tan\frac{90°}{2} = 1$, and thus $v_f = Az_f^{n-1}(1+z_f^2) = 2A$, whence:

$$\Delta t \;=\; \frac{A}{g}\left(\frac{z_f^{n-1}}{n-1} + \frac{z_f^{n+1}}{n+1}\right) \;=\; \frac{v_f}{2g}\left(\frac{1}{n-1} + \frac{1}{n+1}\right) \;=\; \frac{v_f}{g}\,\frac{n}{n^2-1}$$

Now the Δv "expended" for a constant-n burn is just $ng\Delta t$, which gives:

$$\Delta v \;=\; \frac{v_f}{1-1/n^2}\;. \tag{15}$$

Another way of looking at this is to say the gravity *drag* is:

$$\Delta v_{\text{grav}} \;\equiv\; \Delta v - v_f \;=\; \frac{v_f}{n^2-1}\;. \tag{16}$$

This applies equally for descent from a horizontal trajectory, replacing v_f with the *initial* horizontal speed.

Shortcomings

These formulae are inaccurate for speeds approaching orbital speed, as they do not account for the curvature of the horizontal surface, and consequent reduction in effective gravity due to centrifugal force. This means that the required thrust-to-weight ratio n from Equation (11) is an *overestimate* for the initial deorbit burn, but becomes more accurate during the final descent.

The constant-n approximation is also not likely to hold when the vehicle's mass changes significantly over the course of the burn. The vehicle *can* maintain a constant acceleration (by lowering thrust in proportion to weight), but this is not as efficient as allowing higher acceleration. This approximation also neglects air drag, which reduces the maximum achievable acceleation ng during ascent but allows for greater deceleration during descent.

One approach for safe descent is to treat the calculated thrust to weight in Equation (11) as a conservative estimate, firing the motors when it approaches their *current* maximum performance. The calculated n will decrease as the vehicle slows from orbital speed, and the motors' maximum n will increase as the vehicle loses mass: when the discrepancy gets too large one can cut off thrust, wait for the calculated n to approach the new maximum n, then resume deceleration. This is less efficient that a continuous maximum-thrust ("suicide") burn, but it is relatively safe so long as one can tolerate the inefficiency.